# Business 4720 - Class 21

## Reinforcement Learning – Tabular Methods

Joerg Evermann

Faculty of Business Administration
Memorial University of Newfoundland
jevermann@mun.ca

# This Class

## What You Will Learn:

- Reinforcement Learning
  - Multi-Armed Bandits
  - Value Functions
  - Monte-Carlo (MC) Methods
  - Temporal-Difference (TD) Methods

# Based On

Richard S. Sutton and Andrew G. Barto (2018) *Reinforcement Learning – An Introduction*. 2nd edition, The MIT Press, Cambridge, MA. (SB)

`http://incompleteideas.net/book/the-book.html`

Chapters 2–7

# Resources

Implementations are available on the following GitHub repo:
`https://github.com/jevermann/busi4720-rl`

The project can be cloned from this URL:
`https://github.com/jevermann/busi4720-rl.git`

**Online Learning by Acting in an Environment**

### Ideas

- Maximize return
- Immediate and delayed/subsequent rewards
- Discover which actions to take by trying them
- Tradeoff between exploration and exploitation
- Uncertain/random/stochastic environment

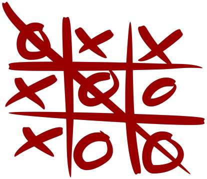
*Problem cannot be tackled by optimization (e.g. dynamic programming), because of incomplete knowledge of environment.*

# Reinforcement Learning

## Core Elements

- **Policy** $\pi$ (probability of taking action $a$ in state $s$)
- **Reward** $R$ (received from the environment after each action)
- **Return** $G$ (possibly discounted sum of future rewards)
- **State value function** $v$ (expected return for each state)
- **Action value function** $q$
(expected return for each state and action)
- **Model** $p$ (behaviour of the environment, optional)

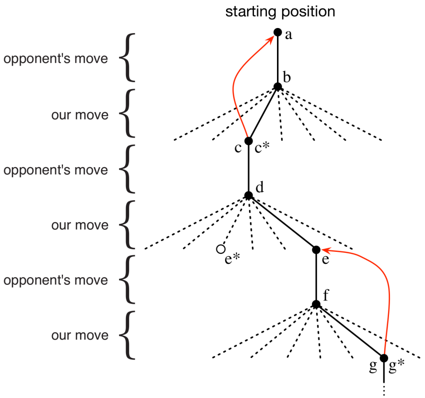
# Introductory Example – Tic-Tac-Toe (Naughts and Crosses)



https://commons.wikimedia.org/wiki/File:Tic_tac_toe.svg

- Value is probability of winning from state
- Value of any state with X-X-X is 1.0
- Value of any state with O-O-O or full board is 0.0
- Initial values are 0.5

# Introductory Example – Tic-Tac-Toe [cont'd]




Source: SB Figure 1.1

- Normally, move *greedily* to highest-valued next state
- Sometimes, make random *exploratory* move
- After each greedy move, update state value towards value of later state with *step size* $\alpha$
- *Temporal-difference* learning $V(S_{t+1}) - V(S_t)$
- Take advantage of information *during* episode/game.

$$V(S_t) \leftarrow V(S_t) + \alpha\left[V(S_{t+1}) - V(S_t)\right]$$

# Example Applications in Business and Management

- **Marketing**: Learn which online ads to show to which site visitor
- **HR**: Learn which employee to assign to which task
- **Operations**: Learn which work item to assign to which machine
- **Logistics**: Learn which item to route on which truck or flight
- ...

# K-armed Bandits


https://commons.wikimedia.org/wiki/File:Antique_one-armed_bandit,_Ventnor,_Isle_of_Wight,_UK.jpg

- Stateless environment
- $k$ possible actions $A_t$ at time $t$ with stochastic reward $R_t$
- Estimate *action value* as:

$$Q_t(a) = \frac{\sum_{i=1}^{t-1} R_i \times \mathbb{1}_a}{\sum_{i=1}^{t-1} \mathbb{1}_a} \qquad \text{(average reward)}$$

- $\epsilon$*-greedy policy*: With probability $\epsilon$ take random action, with probability $1 - \epsilon$ take optimal action

$$A_t = \operatorname*{argmax}_a Q_t(a)$$

- Incremental implementation

$$Q_{t+1}(a) = Q_t(a) + \frac{1}{t}\left[R_t(a) - Q_t(a)\right]$$

MEMORIAL UNIVERSITY

# General Update Rule for Estimates

$$\mathit{NewEstimate} \leftarrow \mathit{OldEstimate} + \mathit{StepSize}\,[\mathit{Target} - \mathit{OldEstimate}]$$

$[\mathit{Target} - \mathit{OldEstimate}]$ is the *error* in the estimate

# K-armed Bandits – Example

## A simple bandit algorithm

Initialize, for $a = 1$ to $k$:

$$Q(a) \leftarrow 0$$
$$N(a) \leftarrow 0$$

Loop forever:

$$A \leftarrow \begin{cases} \text{argmax}_a \, Q(a) & \text{with probability } 1 - \epsilon \\ \text{a random action} & \text{with probability } \epsilon \end{cases}$$
$$R \leftarrow \text{bandit}(A)$$
$$N(A) \leftarrow N(A) + 1$$
$$Q(A) \leftarrow Q(A) + \frac{1}{N(A)} \left[ R - Q(A) \right]$$

# K-armed Bandits – Python

Environment:

```
class k_bandit_env:
    def __init__(self, k):
        self.k = k
        self.mean_rewards = []

        for i in range(self.k):
            self.mean_rewards \
                .append(random.normalvariate(0, 1))

    def step(self, action):
        mean = self.mean_rewards[action]
        reward = random.normalvariate(mean, 1)
        return reward
```

# K-armed Bandits – Python

Agent:

```
class k_bandit_agent:
    def __init__(self, k, epsilon, initial_value):
        self.k = k
        self.epsilon = epsilon
        self.env = k_bandit_env(k)

        self.Q = [initial_value] * self.k
        self.N = [.0] * self.k

    def determine_action(self):
        if random.uniform(0,1) < self.epsilon:
            # explore
            action = random.randint(0, self.k-1)
        else:
            # exploit
            action = self.Q.index(max(self.Q))
        return action
```

# K-armed Bandits – Python

Agent (continued):

```
    def train(self, steps):
        rewards = []
        for step in range(steps):
            action = self.determine_action()
            reward = self.env.step(action)
            self.N[action] += 1
            self.Q[action] = \
               (reward-self.Q[action])/self.N[action]
            rewards.append(reward)
        return rewards
```

Complete implementation at

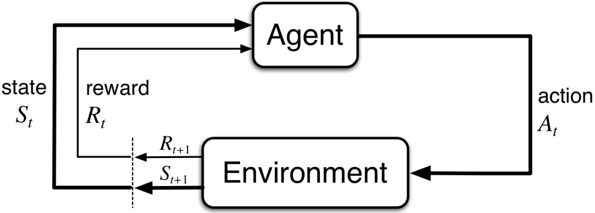
`https://evermann.ca/busi4720/bandits.py`

MEMORIAL UNIVERSITY

# K-armed Bandits – Policy Comparison

Values for $\epsilon$ and the initial $Q$ value affect learning behaviour:

# Markov Decision Processes




Source: SB Figure 3.1

- **Trajectory**:

$$S_0, A_0, R_1, S_1, A_1, R_2, \ldots$$

- **Dynamics**:

$$p(s', r | s, a) = Pr\{S_t = s', R_t = r | S_{t-1} = s, A_{t-1} = a\}$$

- **Discounted future return**:

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

- **State value function** of state $s$ under a policy $\pi$:

$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} | S_t = s\right]$$

- **Action value function** for policy $\pi$:

$$\begin{aligned} q_\pi(s, a) &= \mathbb{E}_\pi[G_t | S_t = s, A_t = a] \\ &= \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} | S_t = s, A_t = a\right] \end{aligned}$$

MEMORIAL UNIVERSITY

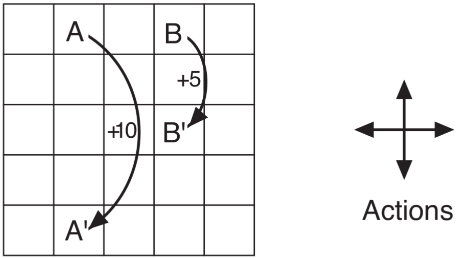
# Bellman Equation

The value function $v$ for policy $\pi$ is the unique solution to the **Bellman equation**:

$$\begin{aligned}
v_\pi(s) &= \mathbb{E}_\pi[G_t | S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} | S_t = s] \\
&= \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r|s, a) \left[ r + \gamma \mathbb{E}_\pi[G_{t+1} | S_{t+1} = s'] \right] \\
&= \sum_a \pi(a|s) \sum_{s', r} p(s', r|s, a) \left[ r + \gamma v_\pi(s') \right] \quad \text{for all } s \in \mathcal{S}
\end{aligned}$$

(Recall: Expected value is sum weighted by probabilities)

- Actions: Up, Down, Left, Right
- Falling off the world results in reward of $-1$
- Other actions result in reward of 0, except for $A$ to $A'$ and $B$ to $B'$ as indicated
- Policy $\pi$ is to take each action with equal probability
- Discount rate $\gamma = 0.9$




| | | | | |
|---|---|---|---|---|
| 3.3 | 8.8 | 4.4 | 5.3 | 1.5 |
| 1.5 | 3.0 | 2.3 | 1.9 | 0.5 |
| 0.1 | 0.7 | 0.7 | 0.4 | -0.4 |
| -1.0 | -0.4 | -0.4 | -0.6 | -1.2 |
| -1.9 | -1.3 | -1.2 | -1.4 | -2.0 |

Source: SB Figure 3.2

# MDP – Optimal Policies

Maximizing the state value function $v$ or action value function $q$ is finding an optimal policy $\pi$:

$$v_*(s) = \max_\pi v_\pi(s)$$
$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

Intuitively, the value of a state under an optimal policy $\pi_*$ is equal to the expected return for the the best action from that state:

$$v_*(s) = \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s, a)$$

MEMORIAL UNIVERSITY

# MDP – Bellman Optimality

$$
\begin{aligned}
v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s, a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t | S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a] \\
&= \max_a \sum_{s', r} p(s', r | s, a)[r + \gamma v_*(s')]
\end{aligned}
$$

Similarly for action value under an optimal policy:

$$
\begin{aligned}
q_*(s, a) &= \mathbb{E}\left[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') | S_t = s, A_t = a\right] \\
&= \sum_{s', r} p(s', r | s, a)[r + \gamma \max_{a'} q_*(s', a')]
\end{aligned}
$$

MEMORIAL UNIVERSITY

# Dynamic Programming

**Intuition:**

1. Start from random value function and policy
2. Compute updated value function (iteratively)
3. Adjust policy based on updated value function
4. Repeat from (2) until convergence

MEMORIAL UNIVERSITY

# Dynamic Programming

## Iterative Policy Evaluation

Loop:

$\Delta \leftarrow 0$

Loop for each $s \in \mathcal{S}$ :

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until $\Delta < \theta$

# Iterative Policy Evaluation in Python

```python
# Initialize value function V
V = dict()
for state in States:
    V[state] = 0
# Initialize policy pi
pi = dict()
for state in States:
    pi[state] = 0

def evaluate_policy():
    while True:
        Delta = 0
        for s in States:
            v = V[s]
            V[s] = exp_reward(s, pi[s])
            Delta = max(Delta, abs(v - V[s]))
        print(Delta)
        if Delta < theta:
            break
```

MEMORIAL UNIVERSITY

## Iterative Policy Improvement

Loop:

  *stable* $\leftarrow$ *true*

  For each $s \in \mathcal{S}$ :

    *old_action* $\leftarrow \pi(s)$

    $\pi(s) \leftarrow \operatorname{argmax}_a \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$

    If *old_action* $\neq \pi(s)$ then *stable* $\leftarrow$ *false*

  If *stable* then

    return $V \approx v_*$ and $\pi \approx \pi_*$

  else

    go to policy evaluation

# Iterative Policy Improvement in Python

```python
def improve_policy():
    stable = True
    for s in States:
        old_action = pi[s]
        max_r = -math.inf
        max_a = None
        for action in Actions:
            r = exp_reward(s, action)
            if r > max_r:
                max_r = r
                max_a = action
        pi[s] = max_a
        if old_action != pi[s]:
            stable = False
    return stable
```

MEMORIAL UNIVERSITY

# Iterative Policy Improvement in Python

```python
stable = False
while not stable:
    evaluate_policy()
    stable = improve_policy()

print("Optimal Policy:")
print(pi)
```

Complete implementation at

`https://evermann.ca/busi4720/jacks.py`

# Example – Jack's Car Rental

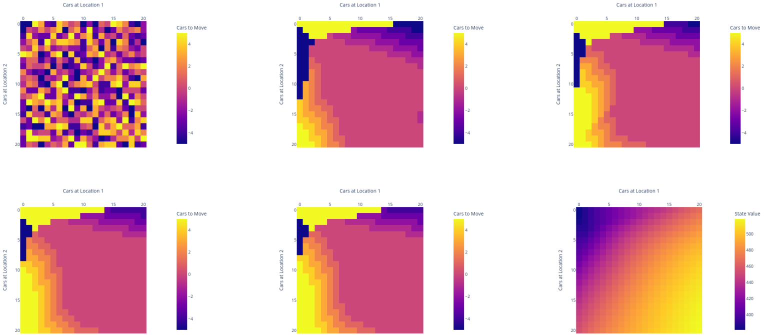
- Jack rents cars at 2 locations with capacity of 20 cars
- Daily rental requests and returns are Poisson distributed
- Move a maximum of 5 cars between locations overnight
- $-2$ reward for each move, $+10$ reward for each satisfied rental request
- How many vehicles to move each night?

```python
States = []
for cars1 in range(21):
    for cars2 in range(21):
        States.append((cars1, cars2))

Actions = range(-5, 5+1)
```

# Example – Policies and Final Value Function

Policies after each improvement, starting with random initial policy. Final state value function at bottom right.



# Monte Carlo Methods

- In practice, dynamics of the environment are not known, i.e. $p(s', r|s, a)$ is unknown; there is *no model* of the environment
- Learning $V$ and $Q$ from *experience*, ie. sample sequences of states, actions, and rewards.
- Consider *episodic tasks*, with a terminal state and finite returns
- Create episodes following policy $\pi$ by interacting with environment
- Similar to the bandit problem, which also learned an action value function, but now with states

# First-visit MC Prediction

- Estimate $V \approx v_\pi$
- Return assigned to state is that of first visit of state

Input: a policy $\pi$ to be evaluated
Initialize:
  $V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$
  $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$
Loop forever (for each episode):
  Generate an episode following $\pi$ : $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$
  $G \leftarrow 0$
  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$ :
    $G \leftarrow \gamma G + R_{t+1}$
    Unless $S_t$ appears in $S_0, S_1, \ldots, S_{t-1}$ :
      Append $G$ to $Returns(S_t)$
      $V(S_t) \leftarrow \text{average}(Returns(S_t))$

# MC Control

- State-value function $V$ not useful without model
- Estimate action-value function $Q$ and $\pi \approx \pi_*$ directly
- To ensure all state-action pairs are visited with a greedy policy, set these as episode starts with some probability (*"exploring starts", ES*)

# MC Control (ES)

Initialize for all $s \in \mathcal{S}, a \in \mathcal{A}$

  $\pi(s) \in \mathcal{A}(s)$ (arbitrarily)

  $Q(s, a) \in \mathbb{R}$ (arbitrarily)

  $Returns(s, a) \leftarrow$ empty list

Loop forever (for each episode):

  Choose $S_0 \in \mathcal{S}, A_0 \in \mathcal{A}(S_0)$ randomly

  Generate an episode following $\pi$ : $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

  $G \leftarrow 0$

  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$ :

    $G \leftarrow \gamma G + R_{t+1}$

    Unless the pair $S_t, A_t$ appears in $S_0, A_0, S_1, A_1, \ldots, S_{t-1}, A_{t-1}$ :

      Append $G$ to $Returns(S_t, A_t)$

      $Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

      $\pi(S_t) \leftarrow \text{argmax}_a\, Q(S_t, a)$

# MC (ES) Control Example – Black Jack

- Card values A, 2, 3, ..., 10
- Ace can be 1 or 11 (”usable ace”)
- Actions: take another card (”hit)” or do not (”stick”)
- Dealer showing initial card, stands on 17 or more
- Over 21 is ”bust” (lost), otherwise closest to 21 wins

# MC (ES) Control Example – Black Jack

Define states $\mathcal{S}$ and actions $\mathcal{A}$:

```
gamma = 1.0
# Define states
States = []
for ace in [0,1]:
    for dealer_showing in range(1,11):
        for hand_sum in range(12, 22):
            States.append((ace,dealer_showing,hand_sum))

# Define actions
Actions = (0, 1)
```

# MC (ES) Control Example – Black Jack

Initialize policy $\pi$, action-value function $Q$, and returns:

```
# Initialize policy
pi = dict()
for s in States:
    pi[s] = random.randint(0,1)

# Initialize action value function
Q = dict()
for s in States:
    for a in Actions:
        Q[(s, a)] = 0

# Initialize returns
Returns = dict()
for s in States:
    for a in Actions:
        Returns[(s, a)] = []
```

MEMORIAL UNIVERSITY

# MC (ES) Control Example – Black Jack

Generate an episode using policy $\pi$ from initial state $S_0$ and initial action $A_0$:

```
def generate_episode(pi, s0, a0):
    terminal = False
    s = s0
    a = a0
    states = [s0]
    actions = [a0]
    rewards = [math.nan]
    while terminal is False:
        sprime, r, terminal = step(s, a)
        rewards.append(r)
        if not terminal:
            aprime = pi[sprime]
            states.append(sprime)
            actions.append(aprime)
            s = sprime
            a = aprime

    return states, actions, rewards, len(rewards)
```

# MC (ES) Control Example – Black Jack

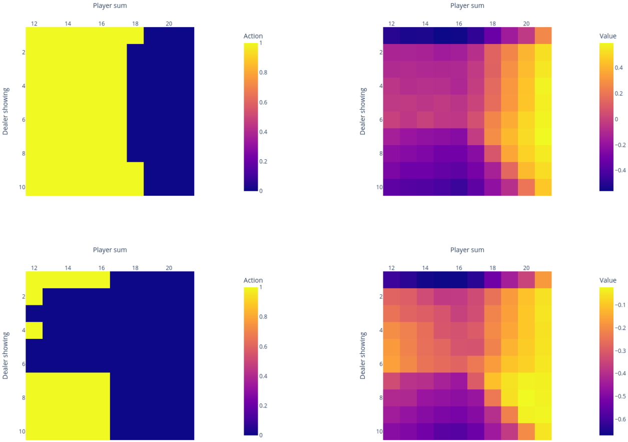
Learn the $Q$ function:

```
for e in range(0, 1000000+1):
    s0 = random.choice(States)
    pi0 = random.choice(Actions)
    S, A, R, T = generate_episode(pi, s0, pi0)
    G = 0
    for t in reversed(range(0, T-1)):
        G = gamma*G + R[t+1]
        if (t == 0) or \
                ((S[t], A[t]) \
                      not in zip(S[0:t-1], A[0:t-1])):
            Returns[(S[t], A[t])].append(G)
            Q[(S[t], A[t])]=mean(Returns[(S[t], A[t])])
            if Q[(S[t], 1)] > Q[(S[t], 0)]:
                pi[S[t]] = 1
            else:
                pi[S[t]] = 0
```

Full example available at
https://evermann.ca/busi4720/blackjack_es.py

# MC (ES) Control Example – Black Jack

Policies and state value function after 1,000,000 episodes.
Usable ace on top, no usable ace at bottom:



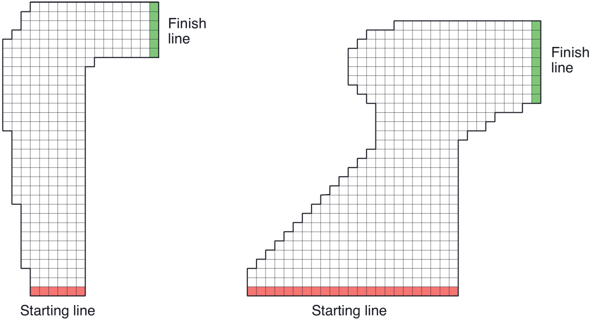
# MC Control Example – Epsilon-Soft Policy

- Exploring starts not always feasible
- Use $\epsilon$-soft policy to ensure exploration (instead of greedy policy)
- Policy $\pi$ is now the probability of taking action *a* in state *s*
- Update $\pi$ with epsilon-soft probabilities, instead of greedy deterministic action

```python
Q[(S[t], A[t])] = mean(Returns[(S[t], A[t])])
# Optimal policy (for two actions)
A_star = 1 if Q[(S[t], 1)] > Q[(S[t], 0)] else 0

for a in Actions:
    if a == A_star:
        pi[(S[t],a)] = 1-epsilon+epsilon/len(Actions)
    else:
        pi[(S[t],a)] = epsilon/len(Actions)
```

# MC Control Example – Racetrack




Source: SB Figure 5.5

- **States**: Position and velocity on race track
- **Actions**: Accelerate +1, 0, -1 in horizontal or vertical dir.
- **Rewards**: -1 for each step, +1 for reaching finish line
- **Noise**: With small probability, actions are ignored

# MC Control Example – Racetrack

```
Actions = []
for y in range(-1, 2):
    for x in range(-1, 2):
        Actions.append((y,x))

Q = dict()
def getQ(s, a):
    if (s, a) not in Q:
        return 0
    else:
        return Q[(s, a)]

pi = dict()
def get_action(s):
    weights = []
    for a in Actions:
        if (s, a) in pi:
            weights.append(pi[(s, a)])
    return random.choices(Actions, weights=weights)[0]
```

# MC Control Example – Racetrack

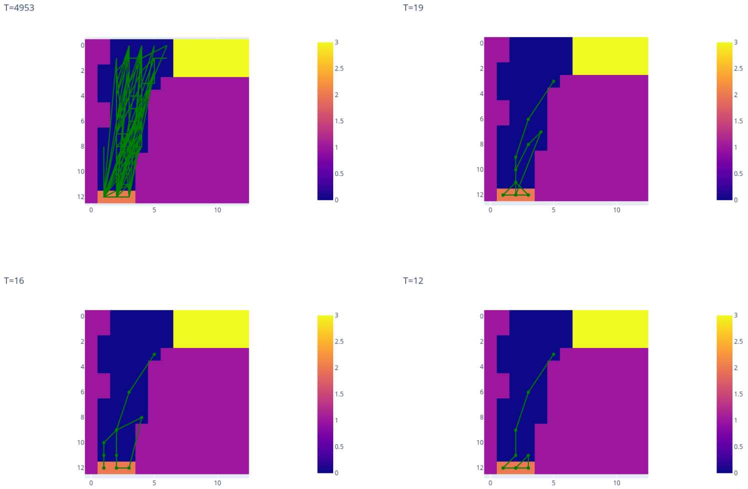
```python
Returns = dict()
def getReturns(s, a):
    if (s, a) not in Returns:
        return []
    else:
        return Returns[(s, a)]
def appendReturn(s, a, r):
    if (s, a) not in Returns:
        Returns[(s, a)] = [r]
    else:
        Returns[(s, a)].append(r)
```

# MC Control Example – Racetrack

```
for e in range(0, 10000+1):
    S, A, R, T = env.generate_episode()
    G = 0
    for t in reversed(range(0, T-1)):
        G = gamma*G + R[t+1]
        if (t == 0) or ((S[t], A[t]) \
                not in zip(S[0:t-1], A[0:t-1])):
            appendReturn(S[t], A[t], G)
            Q[(S[t],A[t])]=mean(getReturns(S[t],A[t]))
            A_star = argmaxQ(S[t])
            for a in Actions:
                if a == A_star:
                    pi[(S[t],a)]=1-eps+eps/len(Actions)
                else:
                    pi[(S[t],a)]=eps/len(Actions)
```

Full example at
https://evermann.ca/busi4720/racetrack.py

Racetrack trajectory after 0, 100, 200, and 10000 learning episodes:



# On-Policy and Off-Policy Learning

## On-Policy Learning

- Trying to learn action values for optimal behaviour
- But have to behave sub-optimal (e.g. $\epsilon$-soft) to explore all actions

## Off-Policy Learning

- Use two policies
- **Target policy** $\pi$: Being learned, typically deterministic, greedy
- **Behaviour policy** $b$: Used to generate behaviour (episodes), typically $\epsilon$-soft
- Behaviour policy $b$ must **cover** target policy $\pi$ (i.e. all possible behaviour under $\pi$ must be generated by $b$)

# Off-Policy MC Control

Initialize for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$ :

$Q(s, a) \in \mathbb{R}$ (arbitrarily)

$C(s, a) \leftarrow 0$

$\pi(s) \leftarrow \mathrm{argmax}_a\, Q(s, a)$

Loop forever (for each episode):

Generate an episode following $b : S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$; $W \leftarrow 1$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$ :

$G \leftarrow \gamma G + R_{t+1}$

$C(S_t, A_t) \leftarrow C(S_t, A_t) + W$

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t, A_t)} [G - Q(S_t, A_t)]$$

$\pi(S_t) \leftarrow \mathrm{argmax}_a\, Q(S_t, a)$

If $A_t \neq \pi(S_t)$ then proceed to next episode

$W \leftarrow W / b(A_t | S_t)$

# Off-Policy MC Control in Python – Policies

## Racetrack example revisited

```python
def b(s):
    weights = []
    for a in Actions:
        if (s, a) in Q:
            weights.append(math.exp(Q[(s, a)]))
        else:
            weights.append(0)
    if len(weights) == 0 or sum(weights) == 0:
        return random.choice(Actions)
    else:
        return random.choices(Actions, weights)[0]

def pi(s):
    a = argmaxQ(s)
    if a is None:
        return random.choice(Actions)
    else:
        return a
```

# Off-Policy MC Control in Python – Policies

## Racetrack example revisited

```
def bprob(a, s):
    if (s, a) not in Q:
        return 1
    weights = []
    for aa in Actions:
        if (s, aa) in Q:
            weights.append(math.exp(Q[(s, aa)]))
    if len(weights) == 0 or sum(weights) == 0:
        return 1
    else:
        return math.exp(Q[(s, a)]) / sum(weights)
```

# Off-Policy MC Control in Python – Learning

## Racetrack example revisited

```
for e in range(0, 10000+1):
    S, A, R, T = env.generate_episode_b()
    G = 0
    W = 1
    for t in reversed(range(0, T-1)):
        G = gamma*G + R[t+1]
        C[(S[t], A[t])] = getC(S[t], A[t]) + W
        Q[(S[t], A[t])] = getQ(S[t], A[t]) + \
                W/getC(S[t], A[t]) * (G-getQ(S[t],A[t]))
        if A[t] != pi(S[t]):
            break
        else:
            W = W * 1/bprob(A[t], S[t])
```

Full example at

https://evermann.ca/busi4720/racetrack_off_policy.py

# Temporal-Difference Learning

## MC Control

- Waits until end of episode before updating $Q$
- Updates based on target (discounted) return $G_t$

$$Q(S_t, a) \leftarrow Q(S_t, a) + \alpha \left[G_t - Q(S_t, a)\right]$$

## TD Control

- Why wait?
- Updates based on target of reward plus discounted future expected return under the optimal action:

$$Q(S_t, a) \leftarrow Q(S_t, a) + \alpha \left[R_{t+1} + \gamma Q(S_{t+1}, a^*_{t+1}) - Q(S_t, a)\right]$$

# On-Policy TD Learning – SARSA

Initialize $Q(s,a)$ for all $s \in \mathcal{S}^+$ , arbitrarily

Loop for each episode:

  Initialize $S$

  Choose $A$ from $S$ using policy derived from $Q$

  Loop for each step of episode:

    Take action $A$, observe $R, S'$

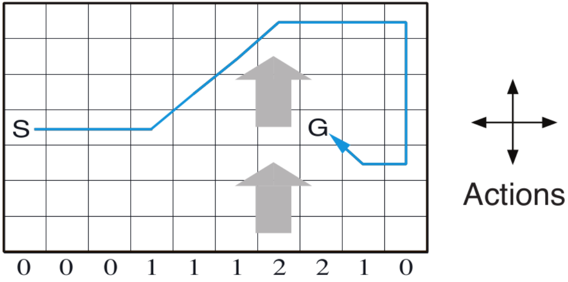
    Choose $A'$ from $S'$ using policy derived from $Q$

    $Q(S,A) \leftarrow Q(S,A) + \alpha\left[R + \gamma Q(S',A') - Q(S,A)\right]$

    $S \leftarrow S'; A \leftarrow A'$

  until $S$ is terminal

# SARSA Example – Windyworld




Source: SB Chapter 6

- Non-discounted ($\gamma = 1$)
- Rewards are -1 until termination
- No penalties for moving off-world

# SARSA Example – Windyworld

```python
# Define states
States = []
for i in range(nrow):
    for j in range(ncol):
        States.append((i, j))
# Define actions
Actions = range(0, 4)
# Initialize Q
Q = dict()
for s in States:
    for a in Actions:
        Q[(s, a)] = random.random()
# Define pi
def pi(s):
    if random.random() < epsilon:
        return random.choice(Actions)
    else:
        return argmaxQ(s)
```

MEMORIAL UNIVERSITY

# SARSA Example – Windyworld

```
for e in range(0, 100):
    terminal = False
    S = windy.reset()
    A = pi(S)
    step = 0
    while terminal is False:
        Sprime, R, terminal = windy.step(A)
        Aprime = pi(Sprime)
        Q[(S,A)] = Q[(S,A)] + alpha*(R + \
            gamma * Q[(Sprime, Aprime)] - Q[(S, A)])
        S = Sprime
        A = Aprime
```

Complete example at `https://evermann.ca/busi4720/windyworld_sarsa.py`

# Generalizing SARSA to n-Step TD Control

**TD Error (1-Step Error)**:

$$\delta_{TD} = R_{t+1} + Q(S_{t+1}, a_{t+1}) - Q(S_t, a)$$

Recall that:

$$Q(S_t, A_t) = \mathbb{E}[G_t | S_t, A_t] \qquad \text{and} \qquad G_t = R_{t+1} + \gamma G_{t+1}$$

**TD Error (2-Step Error)**:

$$\delta_{TD} = R_{t+1} + \gamma R_{t+2} + \gamma^2 Q(S_{t+2}, a_{t+2}) - Q(S_t, a)$$

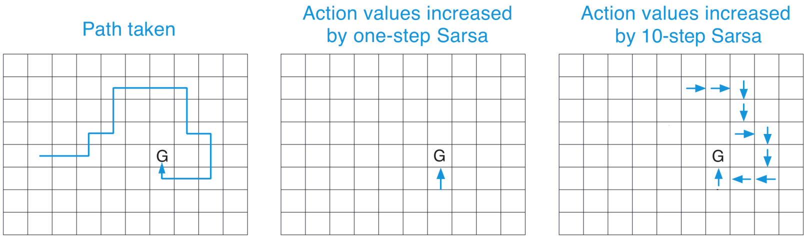
$$\cdots$$

**TD Error (n-Step Error):**

$$\delta_{TD} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n Q(S_{t+n}, a_{t+1}) - Q(S_t, a)$$

MEMORIAL UNIVERSITY

# Generalizing SARSA to n-Step TD Control




Source: SB Figure 7.4

# Off-Policy TD Learning – Q-Learning

Initialize $Q(s, a)$ for all $s \in \mathcal{S}^+$ , arbitrarily

Loop for each episode:

  Initialize $S$

  Loop for each step of episode:

    Choose $A$ from $S$ using policy derived from $Q$

    Take action $A$, observe $R$, $S'$

    $Q(S, A) \leftarrow Q(S, A) + \alpha \left[ R + \gamma \max_a Q(S', A') - Q(S, A) \right]$

    $S \leftarrow S'$

  until $S$ is terminal

MEMORIAL UNIVERSITY

# Q-Learning Example – Windyworld

```
for e in range(0, 1000):
    terminal = False
    S = windy.reset()
    step = 0
    while terminal is False:
        A = pi(S)
        Sprime, R, terminal = windy.step(A)
        Q[(S,A)] = Q[(S,A)] + alpha*(R + \
            gamma * maxQ(Sprime) - Q[(S, A)])
        S = Sprime
```

Complete example at `https://evermann.ca/busi4720/windyworld_q_learning.py`

Steps per episode to termination:

**SARSA**


**Q-Learning**